Minimum-Time Multi-Dimensional Gradient Waveform Design using Convex Optimization

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Rapid sequences demand very efficient use of gradient strength and slew-rate. We present a general method of designing multi-dimensional, minimum-time waveforms, particularly preparatory and rewinder gradients, using standard optimization techniques. This method is easily adapted to design gradients subject to constraints on endpoints, gradient moments, amplitude and slew rate, and is fast enough to design most gradient waveforms during scan prescriptions. We have applied these techniques to design general oblique gradients as well as freely-rotatable rewinder gradients for spiral imaging.

Introduction: Refocused steady-state free precession (SSFP) sequences (TrueFISP, FIESTA or balanced-FFE), demand efficient use of gradients so that high resolution can be achieved while keeping repetition times very short (2-5 ms). Many multi-dimensional gradient design problems are separable into a series of simple one-dimensional designs. Non-separable problems such as the design of freely-rotatable gradients for spiral or oblique scanning quickly become complex as constraints on k-space or higher-order moments are added. Optimal-control solutions as well as 2D-root finding have been used for some of these problems [1,2]. We have expanded the method proposed by Simonetti et al. [3] to develop a very general technique for minimum-time, multi-dimensional gradient waveforms. Our technique produces a minimum-time solution to numerous gradient waveform design problems, usually with computation times of less than a few seconds.

Methods: A gradient waveform can be expressed as a discrete-time sequence of 2D or 3D gradient vector values, $\mathbf{g}_i$, where $i = 1,...,N$ is the time index. For a gradient to be freely rotatable, the Euclidian norm of the gradient and slew rate $\mathbf{s}_i = (g_{ix}, g_{iy}, g_{iz})/\Delta t$ must be within the maximum gradient amplitude and slew rate for an axis ($\Delta t$ is the gradient sampling rate). Both Euclidian constraints can be expressed as quadratic constraints or approximated using piecewise-linear constraints to represent segments of a circle. If the design is for a particular oblique angle, then the limits on $\mathbf{g}_i$ and $\mathbf{s}_i$ can instead be expressed as linear constraints. Other design constraints such as the initial and final gradient vector, the desired gradient area or higher gradient moments can all be expressed as linear constraints on $\mathbf{g}_i$.

Given the above constraints, the design problem can be formulated as a standard convex optimization problem. Efficient techniques have been developed that quickly find whether a feasible solution exists given the constraints, and converge to the optimal solution given a separate cost function. Gradient waveform design can solved using the L1-norm formulation [4], or using second-order cone programming (SOCP) which finds a solution subject to quadratic constraints [5]. For minimum-time gradient design we are only interested in existence of a solution given the constraints and a particular $N$. Assuming the initial or final gradient is zero, then existence of a solution of length $N$ guarantees a solution of length $N+1$. Thus a binary-search of lengths $N$ for which a feasible solution exists determines a minimum-time solution.

As an example, we show the design of a rewinder gradient for spiral SSFP imaging [2]. Given a 2D spiral readout gradient with arbitrary area, the rewinder starts with the same gradient vector. At the end of the rewinder, the gradient, total area and total first moment must all be zero so that all spins with constant velocity are refocused. Our design is for a short (1.7 ms) spiral readout. We assumed a constant gradient limit of 40 mT/m, and a constant maximum slew-rate of 150 T/m/s in the design.

Results and Discussion: Figure 1 shows the spiral readout and minimum-time (1.2 ms) rewinder gradient. This is a 30% time reduction compared to a simple rewinder design that limits each axis to 70% if the maximum to guarantee a freely-rotatable solution. We have also repeated the time-optimal design shown by Dale and Duerk [1] with the same results.

We have tested both L1-norm and SOCP designs. For 3D waveforms or 2D waveform durations less than about 1.5 ms, the SOCP method is quicker, and produces slightly shorter solutions compared to the L1-norm method. However, L1-norm uses more standard software, and is quicker than SOCP for design of longer gradient waveforms. With both methods, the constant-maximum-slew-rate model can be easily replaced by a more accurate voltage model [6], which usually produces even shorter waveforms.

Conclusion: We have developed a multi-dimensional gradient waveform design method that quickly finds minimum-time gradient waveforms for a variety of purposes. This method is particularly useful in the many cases where waveforms cannot be designed analytically, and provides a critical tool for designing waveforms for radial and echo-planar SSFP imaging.

References:

Figure 1. Spiral readout gradient, k-space and first moment with minimum-time rewinder (gray portion) designed using the SOCP method. The x (solid), y (dashed) and magnitude (dotted) of each trajectory are shown.