Chapter 4

Steady-State Sequence Analysis

In periodic MR sequences, the magnetization can be completely predicted given a knowledge of the tissue relaxation parameters, the resonant frequency, and the RF and gradient pulses applied. When the repetition period TR is short, the transverse and longitudinal magnetization from previous excitations significantly affects the current magnetization. After many repetitions of the sequence, the magnetization eventually reaches a *steady state*, where its motion is periodic with the same period as that of the pulse sequence. Such sequences are called steady-state sequences.

Steady-state imaging sequences have great potential in imaging because the short repetition time allows for very high SNR efficiency and also because they tend to provide high contrast between fluids and other tissues. However, these sequences present other challenges that need to be overcome, such as high sensitivity to resonant frequency and an oscillatory signal during the “transient response” as the steady state evolves. Off-resonance results in bands across the image corresponding to different resonant frequency bands [61]. Reduction of transient oscillations is the topic of Chapter 5. However, first a detailed analysis of steady-state sequences is useful, as this provides significant intuition about the way in which steady-state magnetization evolves.
4.1 Signal Calculation for Refocused-SSFP

Perhaps the simplest steady-state sequence is refocused-SSFP or true fast imaging with steady-state precession (true-FISP), a method that has recently regained popularity for clinical use [62–65]. A refocused-SSFP sequence, shown in Fig. 4.1, simply consists of a periodic application of an RF excitation pulse that tips the magnetization through an angle $\alpha$ about the x axis. Gradients are used between $\alpha$ pulses for imaging, but all gradients are fully rewound from one $\alpha$ pulse to the next, and their effects can be ignored in this analysis.

![Figure 4.1: Refocused-SSFP Sequence. RF pulses tip the magnetization through an angle $\alpha$ about the x axis. The points a-d are used to calculate the steady-state magnetization.](image)

There are many prior examples of analysis of the signal in steady-state sequences such as [44,66,67]. The following is a matrix-algebra method similar to that of [67].

To calculate the steady-state signal, we follow the magnetization through one period of this sequence, starting at point a, and continuing to point d. Between points a and b, the magnetization undergoes free precession, and Eq. 2.11 can be used to write

$$M_b = P (TR - TE) M_a + D (TR - TE)$$  \hspace{1cm} (4.1)

The rotation from point b to c is represented using Eq. 2.14 as

$$M_c = R(\alpha, 0) M_b$$  \hspace{1cm} (4.2)
Chapter 4. Steady-State Sequence Analysis

Finally, from point c to point d, Eq. 2.11 is used again to write

\[ M_d = P(TE)M_c + D(TE) \]  \hspace{1cm} (4.3)

Letting \( M = M_a \) and \( M' = M_b \), and combining Eqs. 4.1-4.3 we can express the propagation of magnetization from one period (M) to the next (M') as

\[ M' = AM + B \]  \hspace{1cm} (4.4)

where

\[ A = P(TE)R(\alpha, 0)P(TR - TE) \]  \hspace{1cm} (4.5)

and

\[ B = D(TE) + P(TE)R(\alpha, 0)D(TR - TE) \]  \hspace{1cm} (4.6)

Equation 4.4 is powerful, as it resembles a state-space representation of a linear system. The matrix \( A \) and the vector \( B \) are functions of \( T_1, T_2 \), the resonant frequency, and the applied pulse sequence parameters. We can choose any point, TE, in any periodic sequence and obtain an expression of the form of Eq. 4.4.

The steady-state magnetization is easily obtained by noting that in the steady state the magnetization motion is periodic and \( M' = M = M_{ss} \) and then solving Eq. 4.4 for \( M \):

\[ M_{ss} = (I - A)^{-1} B \]  \hspace{1cm} (4.7)

The above derivations have used a refocused-SSFP sequence as an example. Before examining refocused-SSFP in more detail, we show the usefulness of the form of Eqs. 4.7 and 4.4.

4.2 General Signal Analysis for Steady-State Sequences

The magnetization of any periodic sequence can be expressed in the form of Eqs. 4.7 and 4.4 by deriving the appropriate \( A \) and \( B \) matrices for the particular sequence,
tissue and resonant frequency. This section uses these equations to provide a general analysis of both the transient response and steady-state response of sequences.

### 4.2.1 Transient Response Analysis

A very general transient analysis of magnetization was given in 1955 by Jaynes [5]. This section provides a similar analysis, but focuses on the magnetization dynamics as a discrete-time system.

To examine the transient response, we rewrite Eq. 4.4 as

\[
M_{k+1} = AM_k + B
\]  

(4.8)

where \( M_k \) and \( M_{k+1} \) represent the magnetization vector at a given point in the \( k^{th} \) and \( (k+1)^{th} \) periods of the sequence. The \( 3 \times 3 \) matrix \( A \) and the \( 3 \times 1 \) vector \( B \) are defined as before.

Equation 4.8 describes a “state-space” representation of a third-order discrete-time system. Note here that the “input” \( B \) is constant, and generally is a function of all of the same parameters as the “system matrix” \( A \). Thus, applying control theory to analyze this system has limitations. The magnetization can be viewed as the state of a system, and will go from an initial condition to a steady-state value since the system is always stable. The steady-state solution of Eq. 4.8 was given in Eq. 4.7, and can be rearranged as below:

\[
M_{SS} = AM_{SS} + B
\]  

(4.9)

Now, if defining \( \Delta M_k = M_k - M_{SS} \) as the “transient” magnetization on the \( k^{th} \) period of the sequence, then Eqs. 4.8 and 4.9 can be combined to yield simply

\[
\Delta M_{k+1} = A \Delta M_k
\]  

(4.10)

Separating out the transient and steady-state magnetization components shows that the state-space representation of the transient component is a “zero-input” system. The transient magnetization, \( \Delta M_k \) approaches zero as \( k \) grows large. This, as will be shown, is directly linked to the eigenvalues or poles of the system matrix \( A \).
4.2.2 Eigenvector Analysis of Transient Magnetization

Equation 4.10 is more useful when the eigenvector decomposition or "spectral decomposition" of the matrix $A$ is used to write

$$A = V \Lambda V^{-1}$$  \hspace{1cm} (4.11)

where $\Lambda$ is a $3 \times 3$ diagonal matrix of the eigenvalues $\lambda_i$ of $A$ and the columns of the $3 \times 3$ matrix $V$ are the eigenvectors $v_i$ of $A$.

A positive, real-valued eigenvalue will correspond to a purely exponential transient response, while complex-valued or negative real-valued eigenvalues correspond to decaying oscillatory responses. Because $A$ is a real-valued matrix, any complex eigenvalues and eigenvectors of $A$ will occur in complex-conjugate pairs. Since there are three eigenvalues, at least one must be real-valued, and the corresponding eigenvector must also be real-valued. The magnetization described by Eq. 4.8 is a stable system, implying that all eigenvalues have magnitude less than unity. As an example, the eigenvalues for refocused-SSFP are shown in Fig. 4.2. Using the eigenvector decomposition,

$$\Delta M_k = V \Lambda^k V^{-1} \Delta M_0$$  \hspace{1cm} (4.12)

or

$$\Delta M_k = \beta_1 \lambda_1^k v_1 + \beta_2 \lambda_2^k v_2 + \beta_3 \lambda_3^k v_3$$  \hspace{1cm} (4.13)

where $\beta_i$ are the elements of $\beta$, where $\beta = V^{-1} \Delta M_0$. $\beta$ is a $3 \times 1$ vector that gives the components of $\Delta M_0$ along each eigenvector. Hence $\Delta M_k$ can be expressed as a linear combination of the eigenvectors of $A$. The component of $\Delta M_0$ along a given eigenvector decays according to the corresponding eigenvalue. If $\Delta M_0$ is directed entirely along one eigenvector of $A$ then the entire transient response will be directed completely along the $i^{th}$ eigenvector.

In general, we can assume that only one of the eigenvalues is real-valued. If we denote this eigenvalue $\lambda_r$ and the corresponding eigenvector $v_r$, then when $\Delta M_0$
Figure 4.2: The eigenvalues for $A$-matrix for refocused-SSFP, plotted as the resonant frequency is varied for a flip angle of $60^\circ$ and $TR = 10$ ms. The eigenvalues for 0 Hz and 30 Hz off-resonance are shown by the + and $\times$ marks. The angles of the complex eigenvalues are very sensitive to resonant frequency but not to $T_1$ or $T_2$.

is directed along $v_r$, the transient response is a simple exponential decay, rather than an oscillatory decay. This is a highly desirable condition, as oscillations in the transient response tend to cause serious artifacts in images. Figure 4.3 shows a sample transient response from $M_0$ to $M_{SS}$. The component along $v_r$ decays linearly, while the component orthogonal to $v_r$ decays along a spiral pathway.

The matrix $A$ is the product of rotation and diagonal matrices that represent nutations, precessions and decays. With the approximation that $e^{-\frac{\pi}{T_1}} \approx e^{-\frac{\pi}{T_2}}$, $A$ is a normal matrix, or $A^H A = A A^H$ (where $A^H$ is the hermitian-transpose of the matrix $A$). The inclusion of the diagonal matrix representing relaxation means that the steady-state $A$-matrices are not strictly normal. However, the diagonal elements used to define $P$ in Eq. 2.11 are sufficiently similar that $A$ is almost a normal matrix.
The eigenvectors of normal matrices are always orthogonal [68, 69]. This orthogonality would mean that the oscillatory component of the transient response is always in a plane perpendicular to the real-valued eigenvector, and the path in the plane is a circular (rather than elliptical) spiral. The latter effect means that there is no preferred direction in the plane of oscillation. Furthermore, as Jaynes points out [5], if $e^{-\frac{\tau}{\tau_1}} \approx e^{-\frac{\tau}{\tau_2}}$, then the magnetization perfectly follows the surface of a cone whose axis is the real-valued eigenvector.

A simpler way to see this is that any sequence of rotations can be represented by a single rotation about a particular axis. The eigenvectors of a rotation matrix are always orthogonal, with the real-valued eigenvector directed along the axis of rotation, and complex-valued eigenvectors spanning a plane orthogonal to the axis.
Chapter 4. Steady-State Sequence Analysis

There is (currently) no rigorous proof that there exist 3 linearly independent eigenvectors for the matrix $A$ corresponding to a steady-state sequence. However, if there were only 2 linearly independent eigenvectors of $A$, then there would exist a direction $w$ for which $Aw = 0$. Since magnetization magnitude only changes due to relaxation, this case is impossible in short-TR sequences. Thus it can be assumed that the matrix $A$ has 3 linearly independent eigenvectors and 3 non-zero eigenvalues.

4.2.3 Eigenvector Analysis of Steady-State Magnetization

There is also a strong relationship between the steady-state magnetization and the eigenvalue decomposition of $A$. Substituting Eq. 4.11 into Eq. 4.7, we can write

$$M_{ss} = V(I - \Lambda)^{-1}V^{-1}B. \quad (4.14)$$

The product $V^{-1}B$ transforms $B$ into its components along each eigenvector. The diagonal matrix $(I - \Lambda)^{-1}$ scales each component. Finally, the pre-multiplication by $V$ transforms back to the Cartesian coordinate system. If one of the eigenvalues is much closer to 1 than the others, then the corresponding element of $(I - \Lambda)^{-1}$ will dominate. Assuming there is a reasonable component of $B$ along the eigenvector for the eigenvalue close to 1, the result is that $M_{ss}$ lies almost parallel to the eigenvector.

In steady-state MRI sequences, the matrix $A$ is dominated by rotations, and there is commonly an eigenvalue very close to 1. Thus it is common that the steady-state magnetization direction is almost perfectly aligned with the real-valued eigenvector of the matrix $A$. When $A$ is a rotation matrix, this eigenvector lies along the axis of rotation.
4.3 Steady-State Signal Characteristics for Refocused-SSFP

There are many interesting characteristics of the signal in refocused-SSFP sequences. Initially, assume that the echo time is zero—that is the signal is measured at the point b, immediately after the RF tip in Fig. 4.1. The complex signal is the transverse component of the steady-state magnetization calculated above.

4.3.1 Spectral Response

A dominant characteristic of the refocused-SSFP signal is its sensitivity to resonant frequency \([61]\). Figure 4.4 shows the magnitude and phase of the signal as functions of resonant frequency. The magnitude and phase are both periodic with a period of \(1/\text{TR}\). The signal magnitude peaks at resonant frequencies of \((2n+1)/2\text{TR}\), and has signal nulls at frequencies of \(n/\text{TR}\), where \(n\) takes on integer values. The signal phase consists of fairly linear segments, with a rapid change through \(\pi\) around frequencies of \(n/\text{TR}\). Figure 4.5 shows the “banding artifact” that results from the sensitivity of the signal magnitude to off resonance. This banding artifact has long been a serious obstacle to clinical use of refocused-SSFP sequences, although methods to remove the banding artifact have been suggested \([66,70,71]\).

4.3.2 Flip Angle

The magnitude response is also a function of \(T_1\), \(T_2\) and flip angle. As the flip angle varies from 0 to \(\pi/2\), nulls tend to widen as shown in Fig. 4.6. The magnitude response tends to simply be scaled when \(T_1\) and \(T_2\) vary.
4.3.3 Echo Time

In actual imaging sequences, the echo time, TE, is not zero. Because TE is usually much less than $T_1$ and $T_2$, the magnitude response does not differ significantly from the response for an echo time of zero. However, the phase response differs considerably, as shown in Fig. 4.7. Sharp transitions are still present around frequencies of $n/\text{TR}$. However, the phase response is periodic in the lowest common multiple of $1/\text{TE}$ and $1/\text{TR}$. An important case is that of $\text{TE} = \text{TR}/2$, as this is common in real refocused-SSFP sequences. The phase response has a period of $2/\text{TR}$. More importantly, the phase resembles a square-wave function, with a change of $\pi$ at multiples of $1/\text{TR}$. This feature is desirable in imaging, as for a small range of off-resonant frequencies, it resembles a spin echo and most benefits of spin echoes apply.
Figure 4.5: 2D Axial brain images showing the effect of the magnitude response of Fig. 4.4. (a) Image with banding artifact, visible as a dark band and (b) image with artifact corrected by using a technique with reduced sensitivity to resonant frequency [70].

Figure 4.6: SSFP signal magnitude as a function of frequency for different flip angles. At low flip angles, the "passband" includes a dip, and the signal nulls are very narrow. As the flip angle increases, the signal nulls widen and the signal level peaks in the center of the passband.
Figure 4.7: Refocused-SSFP phase as a function of frequency for $T_1$, $T_2$, and flip angle as in Fig. 4.4, at different echo times. (a) TE=0, (b) TE = TR/4, (c) TE = TR/2 (d) TE= 3/4TR. The phase profile is nearly flat when TE = TR/2.
4.3.4 No-Relaxation Approximation for Refocused-SSFP Direction

The analytic solution for the steady-state magnetization in refocused-SSFP imaging is generally not simple. However, the parameter to which the steady-state magnetization is most sensitive is the resonant frequency. The dependence of the steady-state magnetization direction on resonant frequency can be seen by approximating $\text{TR} \ll T_1$ and $\text{TR} \ll T_2$. In the steady state, the tip due to the RF excitation cancels the off-resonant precession during TR, as shown in Fig. 4.8. The result is that the magnetization directions are approximately related as

$$M_x \approx M_x \sin \left( \frac{\Delta \omega \text{TR}}{2} \right) \cot \left( \frac{\alpha}{2} \right)$$

(4.15)

Possibly the most useful consequence of this analysis is to note again that at time $\text{TR}/2$ the magnetization is entirely within the $x$-$z$ plane. This approximation is quite accurate except for species that have very little off-resonance, as shown in Fig. 4.9, which compares the approximate magnetization direction with the exact direction obtained from Eq. 4.7. The fact that the magnetization is very nearly in a single plane suggests a strategy of trying to duplicate the magnetization at this point in time—a two-dimensional problem.

The idea of neglecting decay to analyze steady-state sequences is effective in the context of eigenvalue analyses too. Without decay, the $A$-matrix is simply a rotation and the real-valued eigenvector of this matrix lies along the axis of rotation.

4.4 Summary

In MR sequences the effects of periodic application of excitation pulses as well as precession and relaxation can be represented using matrix-algebra. A state-space representation can be used to calculate both the transient and steady-state magnetization vector. Eigenvector analysis provides intuition for both the transient
Figure 4.8: Steady-state magnetization progression when $T_1$-decay and $T_2$-decay are neglected. Magnetization vector is shown for four different times during the sequence (a). Position 2 is exactly halfway through the RF excitation pulse. The projection of the magnetization vector onto the $y - z$ (b) and $x - y$ planes are shown. The off-resonant precession is exactly cancelled by the RF excitation. Note that at positions 2 and 4, the magnetization lies in the x-z plane for almost any $\Delta \omega$.

and steady-state responses of a sequence with different tissue species and resonant-frequency species. The direction of the real-valued eigenvector in steady-state sequences corresponds to the direction of a smooth exponential transient response as well as the direction of the steady-state magnetization in most cases.

The transient and steady-state characteristics of refocused-SSFP imaging have been examined. Some understanding of both is important in the design of sequences to catalyze steady state or speed up the transient response in Chapter 5.
Figure 4.9: Comparison of actual steady-state magnetization direction (thick solid line, obtained from Eq. 4.7) with approximate direction (thick dashed line, obtained using Eq. 4.15). The angle between the steady-state magnetization and the z-axis is shown as a function of off-resonance frequency for TR = 10 ms, TE = 5 ms, T1=100 ms, T2=30 ms, and a flip angle of 60°. The thin dotted line shows the magnitude of the M component relative to the magnetization length, indicating that the magnetization is in the x-z plane except for frequencies close to resonance.